

# BIHARMONIC PROPERLY IMMERSED SUBMANIFOLDS IN THE EUCLIDEAN SPACES

KAZUO AKUTAGAWA\* AND SHUN MAETA†

ABSTRACT. We consider a *complete* biharmonic immersed submanifold  $M$  in an Euclidean space  $\mathbb{E}^N$ . Assume that the immersion is *proper*, that is, the preimage of every compact set in  $\mathbb{E}^N$  is also compact in  $M$ . Then, we prove that  $M$  is minimal. It is considered as an affirmative answer to the global version of Chen's conjecture for biharmonic submanifolds.

## 1. Introduction and Main Result

Let  $M$  be an  $n$ -dimensional connected immersed submanifold in the Euclidean  $N$ -space  $\mathbb{E}^N$  ( $n < N$ ) and  $\mathbf{x}$  its position vector field. Then, it is well known that

$$(1) \quad \Delta \mathbf{x} = n\mathbf{H},$$

where  $\Delta$  and  $\mathbf{H}$  denote respectively the (non-positive) Laplace operator and the mean curvature vector field of  $M$ . The above equation shows particularly that  $M$  is minimal, that is,  $\mathbf{H} = 0$  if and only if the isometric immersion  $\mathbf{x} : (M, g) \rightarrow \mathbb{E}^N$  is a harmonic map. Here,  $g$  denotes the induced Riemannian metric on  $M$  from  $\mathbf{x}$ .  $M$  is said to be *biharmonic* if  $\mathbf{H}$  satisfies the following:

$$(2) \quad \Delta \mathbf{H} = \frac{1}{n} \Delta^2 \mathbf{x} = 0.$$

It is obvious that every minimal submanifold is biharmonic. We also note that  $M$  is biharmonic if and only if  $\mathbf{x}$  is a biharmonic map.

For biharmonic submanifolds, there is an interesting problem, namely, Chen's Conjecture (cf. [1]):

**Conjecture 1.** *Any biharmonic submanifold  $M$  in  $\mathbb{E}^N$  is minimal.*

There are many affirmative partial answers to Conjecture 1 (cf. [1, 2, 3, 5, 6, 7]). In particular, there are some complete affirmative answers if  $M$  is one of the following: (a) a curve [5], (b) a surface in  $\mathbb{E}^3$  [1], (c) a hypersurface in  $\mathbb{E}^4$  [6, 7].

On the other hand, since there is no assumption of *completeness* for submanifolds in Conjecture 1, in a sense it is a problem in *local* differential geometry. In this article, we reformulate Conjecture 1 into a problem in *global* differential geometry as the following (cf. [8, 9]):

**Conjecture 2.** *Any complete biharmonic immersed submanifold in  $\mathbb{E}^N$  is minimal.*

---

*Date:* June, 2011.

\* supported in part by the Grants-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 21540097.

† supported in part by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, No. 23-6949.

An immersed submanifold  $M$  in  $\mathbb{E}^N$  is said to be *properly immersed* if the immersion  $M \rightarrow \mathbb{E}^N$  is a proper map. Here, we remark that the properness of the immersion implies the completeness of  $(M, g)$ . Our main result is the following, which gives an affirmative partial answer to Conjecture 2:

**Theorem 1.1.** *Any biharmonic properly immersed submanifold  $M$  in  $\mathbb{E}^N$  is minimal.*

For proving Theorem 1.1, the basic tool is the generalized maximum principle technique developed in Cheng-Yau [4] as follows:

*Let  $(M, g)$  be a complete manifold whose Ricci curvature  $\text{Ric}_g$  is bounded from below. Let  $u$  be a smooth nonnegative function on  $M$ . Assume that there exists a positive constant  $k > 0$  such that*

$$(3) \quad \Delta u \geq ku^2 \quad \text{on } M.$$

*Then,  $u = 0$  on  $M$ .*

The outline of proof of the generalized maximum principle is the following. For a fixed point  $x_0 \in M$  and each large positive constant  $\rho > 0$ , consider the following smooth function

$$f(x) := (\rho^2 - r(x)^2)^2 u(x) \quad \text{for } x \in \overline{B_\rho(x_0)},$$

where  $r(x) := \text{dist}_g(x, x_0)$  and  $\overline{B_\rho(x_0)} := \{x \in M \mid r(x) \leq \rho\}$  denote respectively the distance from  $x_0$  and the closed geodesic ball of radius  $\rho$  centered at  $x_0$ . Then, the inequality (3) implies that

$$f(p) \leq c\rho^3 \quad \text{at a maximum point } p \in B_\rho(x_0) := \{x \in M \mid r(x) < \rho\},$$

and hence

$$(4) \quad u(x) \leq \frac{c\rho^3}{(\rho^2 - r(x)^2)^2} \quad \text{for } x \in B_\rho(x_0).$$

Letting  $\rho \nearrow \infty$  in the above inequality, we then get that  $u = 0$  on  $M$ . Here,  $c > 0$  is a positive constant depending only on  $k$ ,  $\dim M$  and the constant  $\kappa \geq 0$  satisfying  $\text{Ric}_g \geq -\kappa$  on  $M$ . The assumption of Ricci curvature bound from below is necessary for the estimate of  $(\Delta r)(p)$  from above (see [10] for details).

When  $(M, g)$  is a Riemannian immersed submanifold in  $\mathbb{E}^N$ , it is impossible to get such Ricci curvature bound from below without an assumption of boundedness for the second fundamental form  $h$  of  $M$ . However, for Conjecture 2, any assumption for  $h$  is artificial in some sense. To overcome this difficulty, we consider the function

$$F(x) := (\rho^2 - |\mathbf{x}(x)|^2)^2 u(x) \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho})$$

instead of  $f(x)$ , where  $|\mathbf{x}(x)|^2 := \langle \mathbf{x}(x), \mathbf{x}(x) \rangle$  denotes the square-norm of the position vector  $\mathbf{x}(x)$  of  $x \in M$  in  $\mathbb{E}^N$  and  $\overline{\mathbf{B}_\rho} := \{x \in \mathbb{E}^N \mid |\mathbf{x}(x)| \leq \rho\}$ . From the formula (1), we then get

$$|\Delta \mathbf{x}(x)| \leq n|\mathbf{H}(x)|.$$

Moreover if  $M$  is biharmonic, by the harmonicity (2) combined with the above estimate, one can obtain a similar estimate to (4) for  $u(x) := |\mathbf{H}(x)|^2$  especially (see Section 3 for details).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. Section 3 is devoted to the proof of Theorem 1.1.

**Acknowledgements.** The first author would like to thank Reiko Aiyama, Nobumitsu Nakauchi and Hajime Urakawa for helpful discussions. He also would like to thank Luis Alías and Reiko Miyaoka for useful comments.

## 2. Preliminaries

Let  $M$  be an  $n$ -dimensional immersed submanifold in  $\mathbb{E}^N$ ,  $\mathbf{x} : M \rightarrow \mathbb{E}^N$  its immersion and  $g$  its induced Riemannian metric. For simplicity, we often identify  $M$  with its immersed image  $\mathbf{x}(M)$  in every local arguments. Let  $\nabla$  and  $D$  denote respectively the Levi-Civita connections of  $(M, g)$  and  $\mathbb{E}^N = (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$ . For any vector fields  $X, Y \in \mathfrak{X}(M)$ , the Gauss formula is given by

$$D_X Y = \nabla_X Y + h(X, Y),$$

where  $h$  stands for the second fundamental form of  $M$  in  $\mathbb{E}^N$ . For any normal vector field  $\xi$ , the Weingarten map  $A_\xi$  with respect to  $\xi$  is given by

$$D_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where  $\nabla^\perp$  stands for the normal connection of the normal bundle of  $M$  in  $\mathbb{E}^N$ . It is well known that  $h$  and  $A$  are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

For any  $x \in M$ , let  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_N\}$  be an orthonormal basis of  $\mathbb{E}^N$  at  $x$  such that  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $T_x M$ . Then,  $h$  is decomposed as at  $x$

$$h(X, Y) = \sum_{\alpha=n+1}^N h_\alpha(X, Y) e_\alpha.$$

The mean curvature vector  $\mathbf{H}$  of  $M$  at  $x$  is also given by

$$\mathbf{H}(x) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i) = \sum_{\alpha=n+1}^N H_\alpha(x) e_\alpha, \quad H_\alpha(x) := \frac{1}{n} \sum_{i=1}^n h_\alpha(e_i, e_i).$$

It is well known that the necessary and sufficient conditions for  $M$  in  $\mathbb{E}^N$  to be biharmonic, namely  $\Delta \mathbf{H} = 0$ , are the following (cf. [1, 2, 3]):

$$(5) \quad \begin{cases} \Delta^\perp \mathbf{H} - \sum_{i=1}^n h(A_{\mathbf{H}} e_i, e_i) = 0, \\ n \nabla |\mathbf{H}|^2 + 4 \operatorname{trace} A_{\nabla^\perp \mathbf{H}} = 0, \end{cases}$$

where  $\Delta^\perp$  is the (non-positive) Laplace operator associated with the normal connection  $\nabla^\perp$ .

From the first equation of (5), we have the following.

**Lemma 2.1.** *Let  $M = (M, g)$  be a biharmonic immersed submanifold in  $\mathbb{E}^N$ . Then, the following inequality for  $|\mathbf{H}|^2$  holds*

$$(6) \quad \Delta |\mathbf{H}|^2 \geq \frac{2}{n} |\mathbf{H}|^4.$$

*Proof.* Under the above notations, the first equation of (5) implies that, at each  $x \in M$ ,

$$(7) \quad \begin{aligned} \Delta |\mathbf{H}|^2 &= 2 \sum_{i=1}^n \langle \nabla_{e_i}^\perp \mathbf{H}, \nabla_{e_i}^\perp \mathbf{H} \rangle + 2 \langle \Delta^\perp \mathbf{H}, \mathbf{H} \rangle \\ &\geq 2 \sum_{i=1}^n \langle h(A_{\mathbf{H}} e_i, e_i), \mathbf{H} \rangle \\ &= 2 \sum_{i=1}^n \langle A_{\mathbf{H}} e_i, A_{\mathbf{H}} e_i \rangle. \end{aligned}$$

When  $\mathbf{H}(x) \neq 0$ , set  $e_N := \frac{\mathbf{H}(x)}{|\mathbf{H}(x)|}$ . Then,  $\mathbf{H}(x) = H_N(x)e_N$  and  $|\mathbf{H}(x)|^2 = H_N(x)^2$ . From (7), we have at  $x$

$$\begin{aligned} \Delta|\mathbf{H}|^2 &\geq 2 H_N^2 \sum_{i=1}^n \langle A_{e_N} e_i, A_{e_N} e_i \rangle \\ &= 2 |\mathbf{H}|^2 |h_N|_g^2 \\ &\geq \frac{2}{n} |\mathbf{H}|^2 H_N^2 \\ &= \frac{2}{n} |\mathbf{H}|^4. \end{aligned}$$

Even when  $\mathbf{H}(x) = 0$ , the above inequality (6) still holds at  $x$ . This completes the proof.  $\square$

### 3. Proof of Main Theorem

*Proof of Theorem 1.1.* If  $M$  is compact, applying the standard maximum principle to the elliptic inequality (6), we have that  $\mathbf{H} = 0$  on  $M$ . Therefore, we may assume that  $M$  is noncompact. Suppose that  $\mathbf{H}(x_0) \neq 0$  at some point  $x_0 \in M$ . Then, we will lead a contradiction.

Set

$$u(x) := |\mathbf{H}(x)|^2 \quad \text{for } x \in M.$$

For each  $\rho > 0$ , consider the function

$$F(x) = F_\rho(x) := (\rho^2 - |\mathbf{x}(x)|^2)^2 u(x) \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho}).$$

Then, there exists  $\rho_0 > 0$  such that  $x_0 \in \mathbf{x}^{-1}(\mathbf{B}_{\rho_0})$ . For each  $\rho \geq \rho_0$ ,  $F = F_\rho$  is a nonnegative function which is not identically zero on  $M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho})$ . Take any  $\rho \geq \rho_0$  and fix it. Since  $M$  is properly immersed in  $\mathbb{E}^N$ ,  $M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho})$  is compact. By this fact combined with  $F = 0$  on  $M \cap \mathbf{x}^{-1}(\partial \mathbf{B}_\rho)$ , there exists a maximum point  $p \in M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho)$  of  $F = F_\rho$  such that  $F(p) > 0$ . We have  $\nabla F = 0$  at  $p$ , and hence

$$(8) \quad \frac{\nabla u}{u} = \frac{2 \nabla |\mathbf{x}(x)|^2}{\rho^2 - |\mathbf{x}(x)|^2} \quad \text{at } p.$$

We also have that  $\Delta F \leq 0$  at  $p$ . Combining this with (8), we obtain

$$(9) \quad \frac{\Delta u}{u} \leq \frac{6 |\nabla |\mathbf{x}(x)|^2|_g^2}{(\rho^2 - |\mathbf{x}(x)|^2)^2} + \frac{2 \Delta |\mathbf{x}(x)|^2}{\rho^2 - |\mathbf{x}(x)|^2} \quad \text{at } p.$$

From (2), we note

$$(10) \quad \begin{cases} \Delta |\mathbf{x}(x)|^2 = 2 \sum_{i=1}^n |\nabla_{e_i} \mathbf{x}(x)|^2 + 2 \langle \Delta \mathbf{x}(x), \mathbf{x}(x) \rangle \leq 2n + 2n |\mathbf{H}| \cdot |\mathbf{x}(x)|, \\ |\nabla |\mathbf{x}(x)|^2|_g^2 \leq 4n |\mathbf{x}(x)|^2. \end{cases}$$

It then follows from (6), (9) and (10) that

$$u(p) \leq \frac{12n^2 |\mathbf{x}(p)|^2}{(\rho^2 - |\mathbf{x}(p)|^2)^2} + \frac{2n^2 (1 + \sqrt{u(p)} |\mathbf{x}(p)|)}{\rho^2 - |\mathbf{x}(p)|^2},$$

and hence

$$F(p) \leq 12n^2 |\mathbf{x}(p)|^2 + 2n^2 (\rho^2 - |\mathbf{x}(p)|^2) + 2n^2 \sqrt{F(p)} |\mathbf{x}(p)|.$$

Therefore, there exists a positive constant  $c(n) > 0$  depending only on  $n$  such that

$$F(p) \leq c(n) \rho^2.$$

Since  $F(p)$  is the maximum of  $F = F_\rho$ , we have

$$F(x) \leq F(p) \leq c(n)\rho^2 \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\overline{\mathbf{B}_\rho}),$$

and hence

$$(11) \quad |\mathbf{H}(x)|^2 = u(x) \leq \frac{c(n)\rho^2}{(\rho^2 - |\mathbf{x}(x)|^2)^2} \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho) \quad \text{and } \rho \geq \rho_0.$$

Letting  $\rho \nearrow \infty$  in (11) for  $x = x_0$ , we have that

$$|\mathbf{H}(x_0)|^2 = 0.$$

This contradicts our assumption that  $\mathbf{H}(x_0) \neq 0$ . Therefore,  $M$  is minimal.  $\square$

#### REFERENCES

- [1] B.-Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Michigan State University, (1988 version).
- [2] B.-Y. Chen and S. Ishikawa, *Biharmonic surfaces in pseudo-Euclidean spaces*, **Memoirs Fac. Sci., Kyushu Univ., Ser. A** **45** (1991), 323–347.
- [3] B.-Y. Chen and S. Ishikawa, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, **Kyushu J. Math.** **52** (1998), 167–185.
- [4] S.-Y. Cheng and S.-T. Yau, *Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces*, **Ann. of Math.** **104** (1976), 407–419.
- [5] I. Dimitrić, *Submanifolds of  $\mathbb{E}^m$  with harmonic mean curvature vector*, **Bull. Inst. Math. Acad. Sinica** **20** (1992), 53–65.
- [6] Th. Hasanis and Th. Vlachos, *Hypersurfaces in  $\mathbb{E}^4$  with harmonic mean curvature vector field*, **Math. Nachr.** **172** (1995), 145–169.
- [7] F. D. Leuven, *Hypersurfaces of  $\mathbb{E}^4$  with harmonic mean curvature vector*, **Math. Nachr.** **196** (1998), 61–69.
- [8] N. Nakauchi and H. Urakawa, *Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature*, to appear in **Ann. Global Anal. Geom.**
- [9] N. Nakauchi and H. Urakawa, *Biharmonic submanifolds in a Riemannian manifold with non-positive curvature*, preprint.
- [10] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, **Comm. Pure Appl. Math.** **28** (1975), 201–228.

*E-mail address:* akutagawa@math.is.tohoku.ac.jp

DIVISION OF MATHEMATICS, GSIS, TOHOKU UNIVERSITY, SENDAI 980-8579, JAPAN

*E-mail address:* shun.maeta@gmail.com

DIVISION OF MATHEMATICS, GSIS, TOHOKU UNIVERSITY, SENDAI 980-8579, JAPAN